AU - 6244

M.A/M.Sc. III rd (Sem.) Examination 2014

Mathematics Paper III (Optional)

Fuzzy Sets, Fuzzy Logic & Applications Part I

Suggested Solution.

Que 1 (i) If A is a fuzzy set over X, the height of A denoted by h(A) is the largest membership grade obtained by any element in the set X.

Mathematically $h(A) = \sup_{x \in X} A(x)$

(ii) The scalar cardinality of D may be calculated as

$$|D| = \sum_{x \in X} D(x) = \sum_{x=1}^{10} \left(1 - \frac{x}{10}\right) = 4.5$$

(iii) Student's choice

(iv) By definition of t-norm $i(0,0) \geq 0$ while by monotonicity we have $i(0,0) \leq i(0,1) = 0$. ~~Thus~~

Thus $i(0,0) = 0$

(v) A decreasing generator is a continuous and strictly decreasing function f from $[0,1]$ to $\mathbb{R}$ such that $f(1) = 0$

(vi) By simple calculation students are suppose to show (i) $u(a,0) = a$

(ii) $b \leq d \implies u(a,b) \leq u(a,d)$

(iii) $u(a,b) = u(b,a)$ &

(vi) $u(a, u(b,d)) = u(u(a,b), d)$

$$[(iv)\ u(a, u(b,d)) = u(a, b+d-bd)$$
$$= a+b+d-bd-ab-ad+abd$$
$$= a+b-ab+d-ad-bd+abd$$
$$= (a+b-ab)+d-(a+b-ab)d$$
$$= u(a+b-ab, d) = u(u(a,b), d)]$$

(vii) A norm operation h satisfies monotonocity, commutativity and associativity like a t-norm or t-conorm but satisfies $h(0,0)=0$ and $h(1,1)=1$. ~~Here h~~ Here $h: [0,1]\times[0,1] \to [0,1]$.

(viii) Student's choice.

(ix) Let two binary fuzzy relations $P(X,Y)$ and $Q(Y,Z)$ be given. Their standard composition in a fuzzy binary relation $R(X,Z)$, which is also denoted by $P(X,Y) \circ Q(Y,Z)$, defined as:

$$R(X,Z) = \max_{y\in Y} \min [P(X,Y), Q(Y,Z)].$$

(X) Since R is a similarity relation if its domain in $X\times X$ then $R(x,x) = 1 \ \forall x \in X$ so $(x,x) \in {}^{\alpha}R \quad \forall \alpha \in [0,1]$

Further if $(x,y) \in {}^{\alpha}R$ then $R(x,y) \geq \alpha$ so that $R(y,x) \geq \alpha$ and therefore $(y,x) \in {}^{\alpha}R \ \forall \alpha \in [0,1]$

Also if $(x,y) \in {}^{\alpha}R$ and $(y,z) \in {}^{\alpha}R$ we have $R(x,y) \geq \alpha$ and also $R(y,z) \geq \alpha$ but then $R(x,z) \geq \alpha$ which means $(x,z) \in {}^{\alpha}R \quad \forall \alpha \in [0,1]$

Hence ~~(≡,≡)~~ $\forall\, \alpha \in [0,1]$ ${}^{\alpha}R$ is a ~~crat~~ crisp equivalence relation.

Que 2. A student is supposed to cover at least following types of fuzzy sets:

1. ~~Ord~~ Ordinary fuzzy sets
2. Interval - valued fuzzy sets
3. Fuzzy sets of type 2
4. Fuzzy sets of type 3
5. Level 2 fuzzy sets
6. Lattice valued fuzzy sets

He / She may illustrate their possible combinations.

Que 3. ~~II~~ Suggested Soln:

(i) We have for any $x \in X$:

$$\begin{aligned} x \in {}^{\alpha}(A \cap B) &\iff (A \cap B)(x) \geq \alpha \\ &\iff \min(A(x), B(x)) \geq \alpha \\ &\iff A(x) \geq \alpha \text{ and } B(x) \geq \alpha \\ &\iff x \in {}^{\alpha}A \text{ and } x \in {}^{\alpha}B \\ &\iff x \in {}^{\alpha}A \cap {}^{\alpha}B \end{aligned}$$

$$\therefore\ {}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$$

Also

$$\begin{aligned} x \in {}^{\alpha}(A \cup B) &\iff (A \cup B)(x) \geq \alpha \\ &\iff \max(A(x), B(x)) \geq \alpha \\ &\iff A(x) \geq \alpha \text{ or } B(x) \geq \alpha \\ &\iff x \in {}^{\alpha}A \text{ or } x \in {}^{\alpha}B \\ &\iff x \in {}^{\alpha}A \cup {}^{\alpha}B \end{aligned}$$

(ii) Since $\overline{^{(1-\alpha)+}A} = {}^{(1-\alpha)+}\bar{A}$ ... we have

for any $x \in X$

$$x \in {}^{\alpha}(\bar{A}) \iff \bar{A}(x) \geq \alpha$$
$$\iff 1 - A(x) \geq \alpha$$
$$\iff 1 - \alpha \geq A(x)$$
$$\iff x \notin {}^{(1-\alpha)+}A$$
$$\iff x \in \overline{{}^{(1-\alpha)+}A}$$

$$\therefore \quad {}^{\alpha}(\bar{A}) = {}^{(1-\alpha)+}\bar{A}$$

Que4 (i) For any $x \in X$ we have

$$\overline{f^{-1}(B)}(x) = 1 - f^{-1}(B)(x)$$
$$= 1 - B(f(x))$$
$$= \bar{B}(f(x))$$
$$= (f^{-1}(\bar{B}))(x).$$

$$\therefore \quad \overline{f^{-1}(B)} = f^{-1}(\bar{B}).$$

(ii) Let $x \in X$ be any element and let us set $f(x) = y$

Then $(f^{-1}(f(A)))(x) = (f(A))(f(x)) = f(A)(y)$

$= \sup\limits_{z \in X \mid f(z) = y} A(z) \geq A(x)$ as $x \in X$ and $f(x) = y$

This is true $\forall x \in X$ so we conclude that

$$f^{-1}(f(A)) \geq A \quad \text{i.e.} \quad A \subseteq f^{-1}(f(A))$$

as desired.

(iii) Let $y \in Y$ be any element. Here we have two cases $f^{-1}(y) \neq \phi$ and $f^{-1}(y) = \phi$. In first case we have $f(f^{-1}(B))(y)$

$= \sup\limits_{x \in X \mid f(x) = y} f^{-1}(B)(x) = \sup\limits_{x \in X \mid f(x) = y} B(f(x)) = B(y)$, while if $f^{-1}(y) = \phi$ we have $f(f^{-1}(B))y = 0 \leq B(y)$

Hence we have $f(f^{-1}(B)) \subseteq B$.

(iv) For all $y \in Y$ we have

$$y \in {}^{\alpha+}(f(A)) \iff f(A)(y) > \alpha.$$

$$\iff \sup_{x \in X | f(x)=y} A(x) > \alpha.$$

$$\iff \exists\, x_0 \in X \text{ such that } f(x_0) = y \text{ and } A(x_0) >$$

$$\iff \exists\, x_0 \in X \text{ such that } f(x_0) = y \text{ and } x_0 \in {}^{\alpha+}A.$$

$$\iff y \in f({}^{\alpha+}A).$$

Hence ${}^{\alpha+}(f(A)) = f({}^{\alpha+}A)$.

(v) For any $y \in Y$, $y \in f({}^{\alpha}A) \Rightarrow \exists\, x_0 \in {}^{\alpha}A$ with $f(x_0) = y$. Thus $f(A)(y) = \sup_{x \in X | f(x)=y} A(x) \geq A(x_0) \geq \alpha$.

This in turn implies that $y \in {}^{\alpha}(f(A))$

Hence $f({}^{\alpha}A) \subseteq {}^{\alpha}(f(A))$

5 (a) We first show that under given conditions c has an equilibrium point. Let us consider an equation $c(a) - a = b$ where $a \in [0,1]$ and $b \in [-1,1]$. For $a = 0$ we get $c(0) - 0 = 1$ while for $a = 1$ we get $c(1) - 1 = 0 - 1 = -1$. As $c : [0,1] \to [0,1]$ is a continuous function by intermediate value theorem for continuous functions we must have that for each $b \in [-1,1]$ there exist at least one $a \in [0,1]$ such that $c(a) - a = b$. Thus in particular, for $b = 0$ yields that there exists a point say $e_c \in [0,1]$ with $c(e_c) - e_c = 0$. This completes the existance part. Now for

uniqueness suppose $\exists\ a_1 \neq a_2$ in the interval $[0,1]$ with $c(a_1) - a_1 = 0$ and also $c(a_2) - a_2 = 0$. If $a_1 < a_2$ we have $c(a_1) \geq c(a_2)$ & also we have $c(a_1) - a_1 = c(a_2) - a_2$ which is clearly a contradiction. Similarly if $a_1 > a_2$ we again arrive at a contradiction. Hence $a_1 = a_2$ i.e. equilibrium point must be unique.

This completes the proof.

5(b) We first obtain lower bound. For any $a, b \in [0,1]$ we have $u(a,b) \geq u(a,0) = a$ by monotonocity and boundary condition. Again by commutativity and same argument we have $u(a,b) = u(b,a) \geq u(b,0) = b$. So we have $u(a,b) \geq \max(a,b)$ i.e. $\max(a,b) \leq u($

For upper bound we first observe that for $b = 0$, $u(a,b) = a$ and for $a = 0$ $u(a,b) = b$. Also since $u(a,b) \geq \max(a,b)$ and $u(a,b) \in [0,1]$ we have $u(a,1) = u(1,b) = 1$. Now by monotonocity we have $u(a,b) \leq u(a,1) = u(1,b) = 1$ Hence $u(a,b) \leq$ $u_{max}(a,b)$.

6(a) We know that for a fuzzy complement c the dual point d_a of $a \in [0,1]$ is such that $c(d_a) - d_a = a - c(a)$, so if $d_a = c(a)$ we get $c(c(a)) - c(a) = a -$ so that $c(c(a)) = a$ i.e. complement is involutive

Conversely let ~~the~~ $c(c(a)) = a$ i.e. complement be involutive. In the equation $c(d_a) - d_a = a - c(a)$, ~~put~~ replace a by $c(c(a))$ to get a functional equation $c(d_a) - d_a = c(c(a)) - c(c(c(a))) = \cancel{c(c(a)) - c(}$ $c(c(a)) - c$ for d_a whose solution is $d_a = c(a)$. This completes the proof.

Que 6(b) Using the fact that $\forall\, a_i \quad 1 \le i \le n \quad a_i = \max(a_i, 0)$ and equation (1), student may obtain

$$h(a_1, a_2, \ldots, a_n) = \max[h(a_1, 0, 0, \ldots, 0), h(0, a_2, 0, \ldots \ldots h(0, 0, \ldots, 0, a_n)]$$

$$= \max[h_1(a_1),\ h_2(a_2),\ \ldots\ h_n(a_n)] \quad \text{—(3)}$$

Further he/she may observe that h_i is a continuous, non-decreasing function from $[0, 1]$ to $[0, 1]$ with $h_i(0) = 0$. Now set $h_i(1) = w_i$, $1 \le i \le n$ then $h_i([0, 1]) = [0, w_i]$, so for any $a_i \in [0, w_i]$ $\exists$ $b_i \in [0, 1]$ such that $a_i = h_i(b_i)$ and hence $h_i(a_i) = h_i(h_i(b_i)) = h_i(b_i) = a_i$ $= \min(w_i, a_i)$ by (2); and for any $a_i \in (w_i, 1]$ $w_i = h_i(1) = h_i(h_i(1)) = h_i(w_i) \le h_i(a_i) \le h_i(1) = w_i$ as h_i is also nondecreasing. Consequently $h_i(a_i) = w_i$ $= \min(w_i, a_i)$. Using this in (3) we conclude the proof.

Que 7. Using definition we observe that $\forall z \in \mathbb{R}$, we have

$$\mathrm{MIN}[A, \mathrm{MAX}(B,C)](z) = \sup_{z = \min[x, \max(u,v)]} \min[A(x), B(u), C(v)] \quad — (1)$$

& $\mathrm{MAX}[\mathrm{MIN}(A,B), \mathrm{MIN}(A,C)]z$

$$= \sup_{z = \max[\min(m,n), \min(s,t)]} \min[A(m), B(n), A(s), C(t)] \quad — (2)$$

In order to complete the proof we have to show that R.H.S of (1) and (2) are equal. Let

$$E = \{\min[A(x), B(u), C(v)] \mid \min[x, \max(u,v)] = z\}$$

& $F = \{\min[A(m), B(n), A(s), C(t)] \mid \max[\min(m,n), \min(s,t)] = z$.

then to conclude that (1) and (2) are equal it is sufficient to show that $E = F$.

For every $a = \min[A(x), B(u), C(v)]$ such that $\min[x, \max(u,v)] = z$ $\exists$ $m = s = x$, $n = u$ and $t = v$ such that

$$\max[\min(m,n), \min(s,t)] = \max[\min(x,u), \min(x,v)]$$

$$= \min[x, \max(u,v)] = z$$

hence $a = \min[A(x), B(u), A(x), C(v)] = \min[A(m), B(n), A(s), C(t)]$, i.e. $a \in F$. Consequently $E \subseteq F$. Now we show that $E = F$ by showing that for any $b \in F$ $\exists$ $a \in E$ such that $b \leq a$ (which will means $F \subseteq E$).

P.T.O.

For any $b \in F$ $\exists$ $m, n, s,$ and t such that

$\max\,[\min(m,n),\ \min(s,t)] = z$ and

$b = \min\,[A(m), B(n), A(s), C(t)]$.

Therefore we have

$$z = \min\,[\max(s,m),\ \max(s,n),\ \max(t,m),\ \max(t,n)]$$

Let $x = \min\,[\max(s,m),\ \max(s,n),\ \max(t,m)]$ $\cancel{\max(t,n)}$

$u = n$, and $v = t$. Then we have $z = \min\,[x, \max(u,v$

On the other hand, we observe that:

$$\min(s,m) \leq x \leq \max(s,m).$$

As A is convex we have

$$A(x) \geq \min\,[A(\min(s,m)),\ A(\max(s,m))]$$

$$= \min\,[A(s), A(m)]$$

Hence $\exists$ $a = \min\,[A(x), B(u), C(v)]$ with $\min\,[x, \max(u,v$

$= z$ [i.e. $a \in E$] and

$a = \min\,[A(x), B(u), C(v)]$ $\cancel{\text{with } \min\,[x, \max(u,v)] =}$

$$\geq \min\,[A(s), A(m), B(n), C(t)] = b$$

i.e. for any $b \in F$ $\exists$ $a \in E$ such that $b \leq a$

~~wit~~ which means $\sup F \leq \sup E$ is true.

This completes the proof.

Ans 8 (a) We know that for fuzzy relations $P(X,Y)$, $P_j(X,Y)$, $Q(Y,Z)$, $Q_j(Y,Z)$ with j running over some index set J and for standard fuzzy union we have the following two equalities:

$$P \overset{i}{\circ} \left(\bigcup_{j\in J} Q_j\right) = \bigcup_{j\in J} (P \overset{i}{\circ} Q_j) \quad — \quad (1)$$

$$\left(\bigcup_{j\in J}(P_j)\right) \overset{i}{\circ} Q = \bigcup_{j\in J} (P_j \overset{i}{\circ} Q) \quad — \quad (2)$$

where $\overset{i}{\circ}$ means the sup-i composition of fuzzy relations.

So we have

$$R_{T(i)} \overset{i}{\circ} R_{T(i)} = \left(\bigcup_{n=1}^{\infty} R^{(n)}\right) \overset{i}{\circ} \left(\bigcup_{m=1}^{\infty} R^{(m)}\right)$$

$$= \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (R^{(n)} \overset{i}{\circ} R^{(m)})$$

$$= \bigcup_{n,m=1}^{\infty} R^{(m+n)} \subseteq \bigcup_{k=1}^{\infty} R^{(k)} = R_{T(i)}$$

[Here $R^{(2)} = R \overset{i}{\circ} R$ and $R^{(n)}$ for any positive integer n is defined inductively.]

Thus $R_{T(i)}$ is i-transitive.

Suppose furthere that if S is a fuzzy relation which is i-transitive and contains R then $R^{(2)} = R \overset{i}{\circ} R \subseteq S \overset{i}{\circ} S \subseteq S$ so by induction we conclude that $R^{(k)} \subseteq S$ for any positive integer k, but then $R_{T(i)} = \bigcup_{k=1}^{\infty} R^{(k)} \subseteq S$. Thus $R_{T(i)}$ is the smallest i-transitive fuzzy relation containing R. Hence the result is true by virtue of definition of i-transitive closure.

Que 8(b) (i) $\Rightarrow$ (ii). Let $P \overset{i}{\circ} Q \subseteq R$ then $\sup i(P(x,y), Q(y,z)) \le R(x,z)$ so that $i(P(x,y), Q(y,z)) \le R(x,z)$ for $\forall x \in X$ $y \in Y$ and $z \in Z$. As we know $i(a,b) \le d \Leftrightarrow w_i(a,d) \ge b$ $\forall a, b, d \in [0,1]$ so we conclude that $w_i(P(x,y), R(x,z)) \ge Q(y,z)$ i.e. $w_i(P^{-1}(y,x), R(x,z)) \ge Q(y,z)$

so that $\inf_{x \in X} w_i(P^{-1}(y,x), R(x,z)) \ge Q(y,z)$

i.e. $(P^{-1} \overset{w_i}{\circ} R)(y,z) \ge Q(y,z)$

i.e. $P^{-1} \overset{w_i}{\circ} R \ge Q$

i.e. $Q \subseteq P^{-1} \overset{w_i}{\circ} R$.

(ii) $\Rightarrow$ (iii) Let $Q \subseteq P^{-1} \overset{w_i}{\circ} R$ then $\forall y \in Y$ & $\forall z \in Z$ we have

$Q(y,z) \le \inf_{x \in X} w_i(P^{-1}(y,x), R(x,z)) \le w_i(P^{-1}(y,x), R(x,z))$

As we know $a \le b \Rightarrow w_i(a,d) \ge w_i(b,d)$ we get

$w_i(Q(y,z), R(x,z)) \ge w_i[w_i(P^{-1}(y,x), R(x,z)), R(x,z)]$

$\ge P^{-1}(y,x)$ $[\because w_i(w_i(a,b), b) \ge a$ $\forall a, b \in [0,1]]$

$\therefore \inf_{z \in Z} w_i(Q(y,z), R^{-1}(z,x)) \ge P^{-1}(y,x)$ $\forall x, \forall y$

i.e. $(Q \overset{w_i}{\circ} R^{-1})(y,x) \ge P^{-1}(y,x)$ $\forall x \in X$ $\forall y \in Y$

Hence $P^{-1} \subseteq Q \overset{w_i}{\circ} R^{-1}$ but then $P \subseteq (Q \overset{w_i}{\circ} R^{-1})^{-1}$.

(iii) $\Rightarrow$ (i) Let $P \subseteq (Q \overset{w_i}{\circ} R^{-1})^{-1}$ then $\forall x \in X$ & $\forall y \in Y$ we have

$$P(x,y) \le (Q \overset{w_i}{\circ} R^{-1})^{-1}(x,y) = (Q \overset{w_i}{\circ} R^{-1})(y,x)$$

$$= \inf_{z \in Z} w_i [\, Q(y,z),\ R^{-1}(z,x) \,].$$

So that

$$P(x,y) \le w_i [\, Q(y,z),\ R^{-1}(z,x) \,]$$

but then

$$i(Q(y,z),\ P(x,y)) \le R^{-1}(z,x) \quad \left[\begin{array}{l} \because w_i(a,d) \ge b \\ \Leftrightarrow i(a,b) \le d \\ \forall a,b,d \in [0,1] \end{array}\right]$$

i.e. $i(P(x,y),\ Q(y,z)) \le R(x,z)$

As this relation is true for any $x \in X$, $y \in Y$ and $z \in Z$ we have

$$P \overset{i}{\circ} Q \subseteq R.$$

N.B.